

# EXTENDABILITY CONDITIONS FOR RAMSEY NUMBERS AND $p$ -GOODNESS OF GRAPHS

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**ABSTRACT.** In their 2011 paper, Omid and Raeisi give a condition that allows considerable extension of Ramsey numbers. We provide a new condition that is equivalent to the former one and show that the collection of graphs satisfying the latter condition is the set of all trees, enabling a new and elementary computation of the multicolor Ramsey number  $R(T, K_{m_1}, \dots, K_{m_t})$  for trees  $T$ . We also prove that the only connected graphs that are  $p$ -good for all  $p$  are trees. Finally, we develop a bound  $N = N(\ell)$  such that for any connected graph  $H$  of girth  $\ell$ ,  $H$  is not  $p$ -good whenever  $p > N$ .

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## 1. INTRODUCTION

In this note we consider only finite simple connected graphs. For given graphs  $G_1, \dots, G_n$  the *multicolor Ramsey number*  $R(G_1, \dots, G_n)$  is the smallest positive integer  $R$  such that any coloring of the edges of a complete graph  $K_R$  with  $n$  colors must yield, for some  $i$ ,  $1 \leq i \leq n$ , a monochromatic isomorph of  $G_i$  in color  $i$ .

Given a graph  $G = (V, E)$ , we denote the degree of a vertex  $v \in V$  by  $d(v)$ . For a given coloring using the color  $i$  we let  $d_i(v)$  denote the number of edges colored  $i$  incident to  $v$ . We also let  $\delta(G) = \min\{d(v) \mid v \in V\}$  and  $\Delta(G) = \max\{d(v) \mid v \in V\}$ , defining  $\delta_i(G)$  and  $\Delta_i(G)$  similarly. We let  $N(v) = \{u \in V \mid uv \in E\}$  denote the neighborhood of  $v$  in  $G$  and, given a coloring of  $G$ , for any color  $i$  let  $N_i(v) = \{u \in V \mid uv \in E, E \text{ is colored } i\}$ .

The object of this note is to study connected graphs  $H$  such that for all connected graphs  $G$  with  $|G| = |H|$  and for each  $p \geq 3$ ,

$$R(H, K_p) \leq R(G, K_p).$$

We note that an easy construction due to Turán [12] shows that for any connected graph  $H$ ,  $R(H, K_p) \geq (|H| - 1)(p - 1) + 1$ , so for graphs  $H$  satisfying the condition in the previous sentence, we have  $R(H, K_p) = (|H| - 1)(p - 1) + 1$ . Burr and Erdős [3] call  $H$   $p$ -good if  $R(H, K_p) = (|H| - 1)(p - 1) + 1$ . We are interested in graphs which are  $p$ -good for all  $p \geq 3$ .

Omid and Raeisi [9] give a beautiful proof of the following result, which gives an “extension property” for such graphs:

**Theorem 1.1.** *Any collection of graphs  $\{G_1, \dots, G_n\}$  for which  $R(G_1, \dots, G_n, K_p) = (R(G_1, \dots, G_n) - 1)(p - 1) + 1$  also satisfies*

$$R(G_1, \dots, G_n, K_{p_1}, \dots, K_{p_t}) = (R(G_1, \dots, G_n) - 1)(R(K_{p_1}, \dots, K_{p_t}) - 1) + 1.$$

Thus for these special graphs  $H$  we have a considerable extension of our knowledge of certain Ramsey numbers involving  $H$ , modulo knowledge of classical complete-graph Ramsey numbers.

Much of the work on  $p$ -goodness of graphs has focused on fixing  $p$  and showing that all sufficiently large graphs satisfying some condition are  $p$ -good. For instance, Nikiforov [7] shows that  $R(C_n, K_p) =$

2010 *Mathematics Subject Classification.* 05D10.

*Key words and phrases.* complete graph,  $p$ -good graph, Ramsey number, tree.

$(n-1)(p-1)+1$  for  $n \geq 4p+2$ , and Burr and Erdős [3] conjecture that for fixed  $p$  and fixed  $d$ , any sufficiently large graph of edge density at most  $d$  is  $p$ -good. Nikiforov and Rousseau [8] respond to some of Burr and Erdős's conjectures by developing a class of graphs, called *degenerate and crumbling*, every sufficiently large member of which is  $p$ -good for a fixed  $p$ . In this note, we reverse the perspective: we fix the graph  $H$  and ask for which  $p$  it is the case that  $H$  is  $p$ -good.

## 2. TWO CONDITIONS

We will compare two conditions that each allow easy computation of certain Ramsey numbers.

- (1) We say that the connected graph  $H$ ,  $|V(H)| = n$ , satisfies the *embedding condition* (Emb) if for all graphs  $G$  such that  $\delta(G) \geq n-1$ ,  $H$  embeds in  $G$ .
- (2) We say that the connected graph  $H$ ,  $|V(H)| = n$ , satisfies the *Omid-Raeisi condition* (OR) if  $R(H, K_p) = (n-1)(p-1)+1$  for all complete graphs  $K_p$ ,  $p \geq 3$ , i.e.,  $H$  is  $p$ -good for all  $p$ .

**Theorem 2.1.** *If  $H$  satisfies (Emb) then it also satisfies (OR).*

*Proof.* Suppose that a connected graph  $H$ ,  $|H| = n$ , satisfies (Emb). Our note above implies we only need show  $R(H, K_m) \leq (n-1)(m-1)+1$ .

We proceed by induction on  $n$ , with base case  $m = 3$ . Let  $N = 2(n-1)+1$  and color the edges of  $K_N$  red and blue ( $r$  and  $b$ ). If there is a vertex  $v \in V(K_N)$  such that  $d_b(v) \geq n$  then any blue edge induced by  $N_b(v)$  yields a blue  $K_3$ , so may assume all edges induced by  $N_b(v)$  are red. This gives a red copy of  $K_n$ , which must contain a red copy of  $H$ . Thus we may assume that  $\Delta_b(K_N) \leq n-1$ , so  $\delta_r(K_N) \geq 2(n-1)-n-1 = n-1$ , so the red edges of  $K_N$  give a graph which must contain a copy of  $H$ , since  $H$  satisfies (Emb).

Assuming we've established the result for a given  $p$ , let us suppose  $N = (n-1)(p+1-1)+1 = p(n-1)+1$  and once again color  $K_N$  red and blue. If  $\delta_r(G) \geq n-1$ , (Emb) again gives us a red copy of  $H$ . Thus we suppose  $\delta_r(K_N) \leq n-2$ , so that  $\Delta_b(K_N) \geq p(n-1)-(n-2) = (n-1)(p-1)+1$ . Thus for some vertex  $v$ ,  $d_b(v) \geq (n-1)(p-1)+1$  and by the induction hypothesis the blue neighborhood  $N_b(v)$  induces a graph containing either a red  $H$  or a blue  $K_{p-1}$ . In the former case we are done and in the latter case this copy of  $K_{p-1}$ , along with  $v$ , forms the blue  $K_p$  needed, and we are done.  $\square$

**Theorem 2.2.** *Let  $H$  be a simple connected graph. Then  $H$  satisfies (Emb) if and only if  $H$  is a tree.*

*Proof.* Let  $|H| = n$  and first suppose  $H$  is not a tree, and therefore contains a cycle. Let  $c$  denote the maximum length of a cycle in  $H$ . By Bollobás (see [2]) there exists a graph  $G$  such that  $\delta(G) \geq n-1$  and with girth at least  $c+1$ .  $H$  cannot embed in such a graph, showing that (Emb) does not hold.

Now suppose  $H$  is a tree and that  $G$  satisfies  $\delta(G) \geq n-1$ . We prove a stronger condition than (Emb), namely that given  $u \in V(H)$  and  $v \in V(G)$  there exists an embedding of  $H$  into  $G$  such that  $u \mapsto v$ . We prove this by induction, the base case  $n = 2$  being trivial. Assume the result for a given  $n$  and let  $|H| = n+1$  and  $G$  such that  $\delta(G) \geq n = (n+1)-1$ . Let  $H' = H \setminus \{w\}$  for some leaf  $w \neq u$ . Let  $N(w) = \{w'\}$ . By inductive hypothesis we may embed  $H'$  in  $G$ , taking  $u$  to  $v$  and  $w'$  to  $x$  for some  $x \in V(G)$ . Since  $d(x) \geq n$  and  $|N(w') \setminus \{w'\}| \leq n-1$ , after embedding  $H'$  in  $G$  at least one vertex remains in  $N(x)$  to which we may map  $w$ , finishing our embedding and our proof.  $\square$

Together our results give us an elementary proof of an already-known fact:

**Corollary 2.3.** *Let  $T$  be a tree,  $|T| = n$ . Then for any  $m_1, \dots, m_t$ ,*

$$R(T, K_{m_1}, \dots, K_{m_t}) = (n-1)R(K_{m_1}, \dots, K_{m_t}) + 1.$$

*Proof.* Combine Theorems 2.1 and 2.2 with Theorem 2.1 from [9].  $\square$

We note that the situation for nonconnected graphs is quite different, for the Turán construction does not yield the desired inequality  $R(H, K_p) \geq (|H|-1)(p-1)+1$  in this case. For a forest  $F$ , in fact, Stahl

[11] proved

$$R(F, K_p) = \max_{i \leq j \leq m} \left\{ (j-1)(p-2) + \sum_{i=j}^m i \cdot k_i \right\},$$

where  $k_i$  is the number of components of  $F$  of order  $i$  and  $m$  is the largest order of a component of  $F$ .

Erdős, Faudree, Rousseau, and Schelp conjecture [5] that if  $n \geq p > 3$ , then the  $n$ -cycle  $C_n$  is  $p$ -good. The condition  $n \geq p$  in their conjecture accounts for the observation that having a cycle that is small relative to  $p$  might prevent a graph from being  $p$ -good; moreover, having any cycle *at all* might prevent a graph from satisfying (OR). This turns out to be true; if  $H$  has a cycle, then  $R(H, K_p)$  grows faster than any linear function of  $p$ .

**Theorem 2.4.** *Suppose a connected graph  $H$  satisfies (OR), i.e.,  $H$  is  $p$ -good for all  $p$ . Then  $H$  is a tree. Thus (Emb) and (OR) are equivalent.*

*Proof.* The proof will proceed by contraposition. Consider the standard random graph model  $G_{n,r}$ , where  $G_{n,r}$  has  $n$  vertices and edges are put in  $G$  with probability  $r$ . We construct such a graph with no cycles of length  $\leq \ell$  and no independent set of size  $\geq p$ , with  $p$  to be chosen later (as a function of  $n$ ). We follow the development in [6].

Choose  $\lambda \in (0, \frac{1}{\ell})$ , and let  $r = n^{\lambda-1}$ . Now, the probability of getting more than  $\frac{n}{2}$  cycles of length  $\leq \ell$  is bounded above by

$$(1) \quad \frac{2n^{\lambda\ell-1}}{1 - n^{-\lambda}},$$

which can be made less than  $\frac{1}{2}$ . The probability of an independent set of size  $\geq p$  is bounded above by

$$\begin{aligned} \binom{n}{p} (1-r)^{\binom{p}{2}} &\leq n^p \exp\left(-\frac{3}{2}(p-1) \log n\right) \\ &= n^{\frac{3-p}{2}}, \end{aligned}$$

which can also be made less than  $\frac{1}{2}$ . We can delete one vertex in each short cycle. Therefore there is a graph on  $n/2$  vertices (for  $n$  sufficiently large) with no cycle of length  $\leq \ell$  and no independent set of size  $\geq p$ . Therefore  $R(C_\ell, K_p) > n/2$ . This so far is standard; what we need is the dependence of  $p$  upon  $n$ . We set  $p = 3n^{1-\lambda} \log n$ . So consider a graph  $H$  whose shortest cycle has length  $\ell$ . Then

$$\begin{aligned} \frac{R(H, K_p)}{p} &\geq \frac{R(C_\ell, K_p)}{p} > \frac{n}{2p} \\ &= \frac{n}{6n^{1-\lambda} \log n} \\ &= \frac{n^\lambda}{6 \log n} \rightarrow \infty \text{ as } p \rightarrow \infty \text{ (and hence } n \rightarrow \infty). \end{aligned}$$

Therefore  $R(H, K_p)$  is superlinear in  $p$ , and so for large enough  $p$ ,  $H$  is not  $p$ -good. □

In fact, we can obtain a lower bound for  $p$  which depends on the order and girth of  $H$ :

**Theorem 2.5.** *Let  $H$  be a connected graph of girth  $\ell$ , with  $h = |H|$ . Then if  $p \geq 36h\ell^4 \exp(12h\ell^4)$ ,  $H$  is not  $p$ -good.*

*Proof.* For  $p$ -goodness to fail, it suffices to show  $\frac{R(H, K_p)}{p} > h$ . From the proof of Theorem 2.4 we have  $\frac{R(H, K_p)}{p} > \frac{n^\lambda}{6 \log n}$  for  $p = 3n^{1-\lambda} \log n$ . We are therefore done if

$$n^\lambda > 6h \log n,$$

which is equivalent to

$$e^{\lambda x} > 6hx$$

after setting  $n = e^x$ . Since  $e^{\lambda x} > 1 + \lambda x + \frac{1}{2}\lambda^2 x^2$ ,  $p$ -goodness will fail if we have

$$1 + \lambda x + \frac{1}{2}\lambda^2 x^2 > 6hx.$$

This inequality holds when  $x > 12h\lambda^{-2}$ , so that  $n = \exp(12h\lambda^{-2})$ .

We must choose  $\lambda \in (0, \frac{1}{\ell})$ , and it will suffice to choose  $\lambda = \ell^{-2}$  to ensure that (1) is less than  $1/2$  for  $n$  this large. Then

$$\begin{aligned} p &= 3n^{1-\lambda} \log n \\ &= 36h\ell^4 \exp(12h\ell^4). \end{aligned}$$

Therefore if  $p \geq 36h\ell^4 \exp(12h\ell^4)$ ,  $p$ -goodness fails.  $\square$

### 3. GOODNESS OF SMALL GRAPHS

**Definition 3.1.** Let  $H$  be a graph not satisfying (OR), i.e.  $H$  is not a tree. Then the goodness of  $H$  is the maximum  $p$  such that  $H$  is  $p$ -good.

In this section, let  $H_1$  stand for  $K_3$  with a pendant edge; let  $H_2$  stand for  $H_1$  with a pendant edge; let  $H_3$  stand for  $C_4$  with a pendant edge; let  $H_4$  stand for  $K_4$  with a pendant edge; let  $H_5$  stand for  $H_4$  with a pendant edge; let  $H_6$  stand for  $K_4 - e$  with a pendant edge; let  $H_7$  stand for  $H_6$  with a pendant edge.

**Proposition 3.2.** The goodness of  $H_1$  is 4.

*Proof.* We may prove that  $R(H_1, K_3) = 7$ . This is a bit of case analysis, but the proof is straight-forward so we omit it. It also follows from the following equality from [3]:

If  $G$  is a connected graph on  $n - 1$  vertices, and  $G_1$  is formed by adding to  $G$  a pendant edge, then

$$(2) \quad R(G_1, K_p) = \max\{R(G, K_p), R(G_1, K_{p-1}) + n - 1\}.$$

Setting  $p = 4$  and  $H_1 = G$ , we get

$$\begin{aligned} R(H_1, K_4) &= \max\{R(3, 4), R(H_1, K_3) + 3\} \\ &= \max\{9, 7 + 3\} \\ &= 10. \end{aligned}$$

Now  $10 = 3 \cdot 3 + 1 = (|H_1| - 1)(p - 1) + 1$ , so  $H_1$  is 4-good. However,  $H_1$  fails to be 5-good. To see this, let  $p = 5$ . Then

$$\begin{aligned} R(H_1, K_5) &= \max\{R(3, 5), R(H_1, K_4) + 3\} \\ &= 14, \end{aligned}$$

which is one more than would be the case if  $H_1$  were 5-good.

Therefore the goodness of  $H_1$  is 4.  $\square$

In Table 1 below, we use equation (2) along with known Ramsey numbers from [10] to compute the goodness of several graphs on 4 or 5 vertices having pendant edges. The numbers in parentheses refer to what the Ramsey number would need to be for that graph to be  $p$ -good for that particular value of  $p$ . The arguments are similar to those used in the proof of Proposition 3.2.

	$K_3$	$H_1$	$H_2$	$C_4$	$H_3$	$K_4$	$H_4$	$H_5$	$K_4 - e$	$H_6$	$H_7$
$R(\_, K_3)$	6	7	9	7	9	9	9	11	7	9	11
$R(\_, K_4)$	9	10	13	10	10	18	18 (13)	18 (16)	11	13	16
$R(\_, K_5)$	14	14 (13)	17	14	14	25	25		16	17	21
$R(\_, K_6)$	18	18	21	18	18	35-41			21	21	26
$R(\_, K_7)$	23	23	25	22	22				28-31	$\geq 28$ (25)	31
$R(\_, K_8)$	28	28	29	26	26						
$R(\_, K_9)$	36	36	36 (33)	30-32	32						
$R(\_, K_{10})$	40-43	34-39	$\geq 36$								
goodness	2	4	8	4	$\geq 9$	2	3	3	3	6	$\geq 7$

TABLE 1. Goodness of some small graphs

## 4. OPEN PROBLEMS

- i. Determine the relationship between goodness and other graph parameters. In light of Theorem 2.4 and the Erdős-Faudree-Rousseau-Schelp conjecture, one might think that girth might be crucial, but the data in Table 1 suggest that the relationship is not simple. For instance, the 4-cycle  $C_4$  has goodness 4, but adding just a single pendant edge increases the goodness by at least 5.
- ii. Say that a collection of graphs  $\{G_1, \dots, G_n\}$  is  $p$ -good if  $R(G_1, \dots, G_n, K_p) = (R(G_1, \dots, G_n) - 1)(p - 1) + 1$ . Is it true that if a collection is  $p$ -good for all  $p$ , then the collection consists only of trees? As far as we know, only collections of stars are known to be  $p$ -good for all  $p$  (see [1]).
- iii. In light of this new definition of goodness, we may restate the Erdős-Faudree-Rousseau-Schelp conjecture as follows:

**Conjecture 4.1.** *For all  $n \geq 4$ , the goodness of  $C_n$  is  $n$ .*

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